Announcements

1) Math colloquium talk tomorrow, 3-4, 2070 CB on Math Circles (middle school outreach)

2) Today is the last day to drop without penalty!

Notation: (intersection, union complement)

If S, T are sets

"$S \cup T$", the union of S and T, is the set whose elements are either in S or in T.

"$S \cap T$", the intersection of S and T, is the set whose elements are in both S and T.

"S^c" is the complement of S, the set of all elements not in S.

Proposition (intersections)

Let W_1 and W_2 be subspaces of a vector space V (over a field $\mathbb{F}$). Then $W_1 \cap W_2$ is also a subspace of V.

Proof: Use the subspace test.

1) $\underline{0_V \in W_1 \cap W_2}$

W_1 and W_2 are subspaces, so $0_V \in W_1$ and $0_V \in W_2$, hence $0_V \in W_1 \cap W_2$.

2) $x-y \in W_1 \cap W_2$ whenever $x, y \in W_1 \cap W_2$

Since $x, y \in W_1 \cap W_2$, $x, y \in W_i$ for $i=1,2$.

Then since W_i is a subspace for $i=1,2$, $x-y \in W_i$ for $i=1,2$, which implies $x-y \in W_1 \cap W_2$.

3) $\alpha x \in W_1 \cap W_2$ whenever $\alpha \in \mathbb{F}$ and $x \in W_1 \cap W_2$

Since $x \in W_1 \cap W_2$, $x \in W_i$ for $i = 1, 2$. Then since W_i is a subspace for $i = 1, 2$, $\alpha x \in W_i$ for $i = 1, 2$, which implies $\alpha x \in W_1 \cap W_2$

Then by the subspace test, $W_1 \cap W_2$ is a subspace of V. $\square$

Scholium: The intersection of two subspaces is nonempty.

<u>Example 1</u>: Let $V = M_n(\mathbb{F})$ over $\mathbb{F}$. Let W_1 be the subspace of all upper triangular matrices. Let W_2 be the subspace of all lower triangular triangular matrices.

$W_1 \cap W_2$ is a subspace by the proposition, and $W_1 \cap W_2$ is all matrices $(\alpha_{i,j})_{i,j=1}^{n}$ with $\alpha_{i,j} = 0_{\mathbb{F}}$ for $i \neq j$, the diagonal matrices.

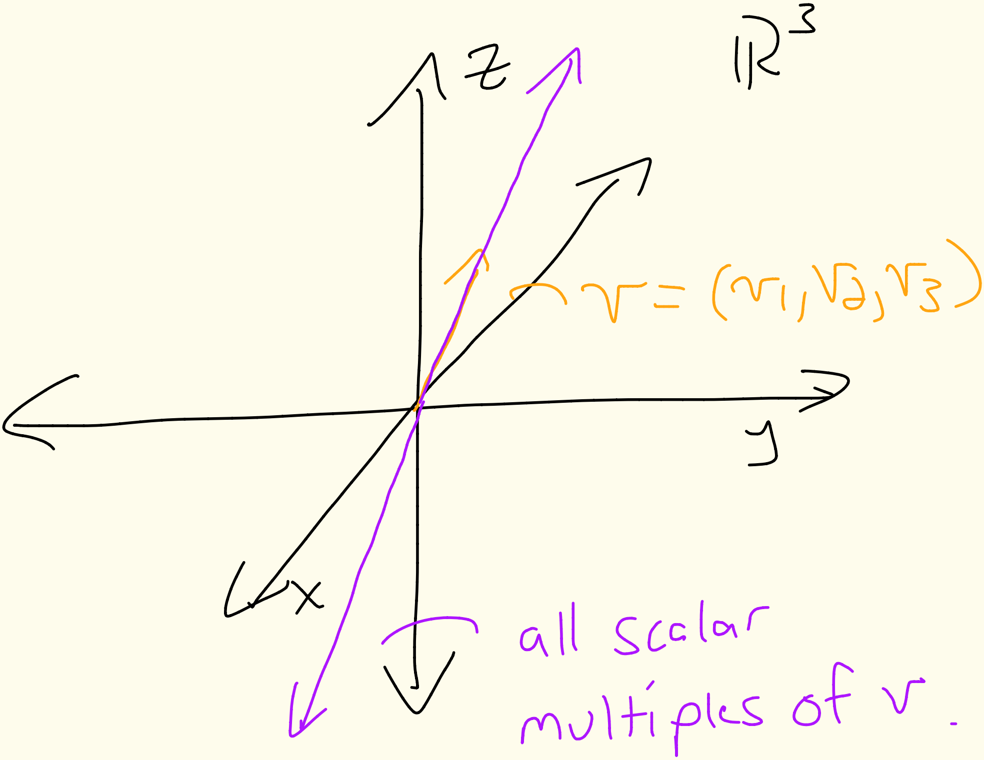
Linear Combinations and Span (Section 1.4)

Think of $\mathbb{R}^3$

(as a vector space over $\mathbb{R}$).

Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$,

suppose $v \neq (0,0,0)$.



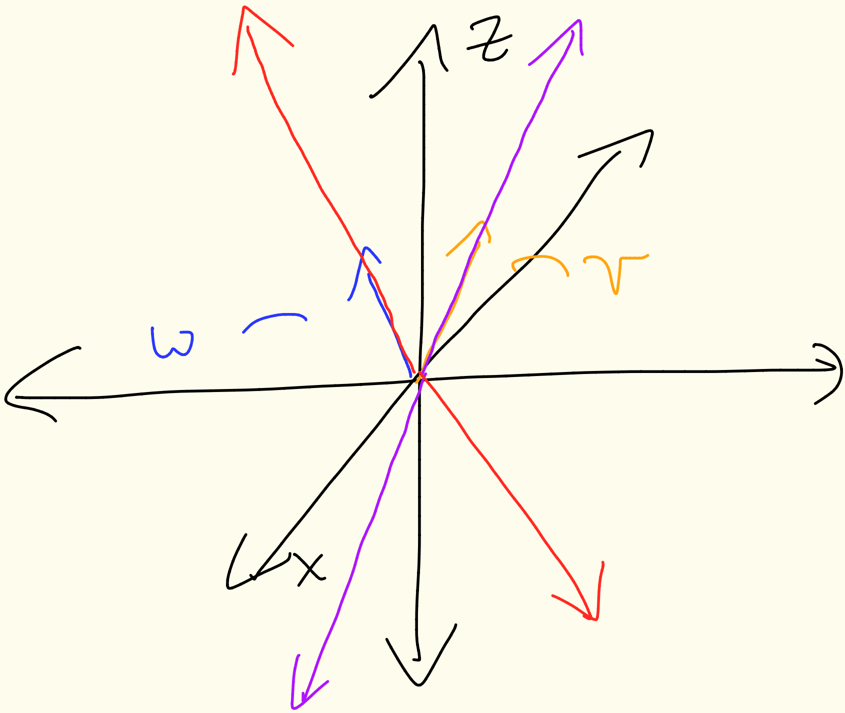
Consider all scalar multiples of v, which includes adding v to itself. You get a line.

Let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_2)$ in $\mathbb{R}^3$,

suppose v is not a scalar multiple of w and that $w \neq (0,0,0)$.

Consider all scalar multiples of v and all scalar multiples of w.

Picture



Consider all sums of scalar multiples of either w or v. You get a plane.

Note that the plane contains the zero vector. This is a subspace of $\mathbb{R}^3$. Similarly, the line determined by w or v is also a subspace of $\mathbb{R}^3$.

Definition: (linear combination)

Start with a vector space V over $\mathbb{F}$ and let S be a nonempty subset of V. A linear combination of elements of S is any vector of the form

$$\boxed{v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n}$$

Where $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}$

and $v_1, v_2, \ldots, v_n \in S$.

For example,

if $S = \{w, v\}$ where

$w = (1,3)$ and $v = (2,-1)$

are vectors in $\mathbb{R}^2$

(considered as a vector

space over $\mathbb{R}$),

then $(2, 13)$

$= 4w - v$

is a linear combination of

v and w.

If, however, we consider $\mathbb{R}^2$ as a vector space over $\mathbb{Q}$, then $(2, 13)$ is still a linear combination of w and v, but $(2\sqrt{2}, -\sqrt{2})$ is not!

Moral: Remember your field!

Definition: (span)

Let V be a vector space over $\mathbb{F}$. Let $S \subseteq V$, $S \neq \{\emptyset\}$. The span (or $\mathbb{F}$-span) is the set of all linear combinations of elements of S.

Proposition: Let V be a vector space over $\mathbb{F}$ and let S be a nonempty subset of V. Then the span of S, denoted by $\text{span}(S)$, is a subspace of V.

proof: Use the subspace test.

1) $\underline{0_V \in \mathrm{Span}(S)}$

Since $S \neq \{\emptyset\}$, $\exists\, w \in S$.

Thm 1.2, p. 12

$0_{\mathbb{F}} \cdot w = 0_V$, and

Since this is a linear combination of elements of S, $0_V \in \mathrm{Span}(S)$.

2) If $x, y \in \text{Span}(S)$, then $x - y \in \text{Span}(S)$.

$x \in \text{Span}(S)$, so

$$x = \sum_{i=1}^{n} \alpha_i w_i$$ for

some α_i, $1 \leq i \leq n$, in $\mathbb{F}$ and

w_i, $1 \leq i \leq n$, in S.

Similarly, $$y = \sum_{j=1}^{m} \beta_j v_j$$

for $\beta_j \in \mathbb{F}$, $v_j \in S$

for all $1 \leq j \leq n$.

Then $x - y$

$$= \sum_{i=1}^{n} \alpha_i w_i - \sum_{j=1}^{m} \beta_j v_j$$

$$= \sum_{i=1}^{n} \alpha_i w_i + \sum_{j=1}^{m} (-\beta_j) v_j .$$

$\in$ Span(S) by definition

3) If $w \in \text{Span}(S)$ and $\alpha \in \mathbb{F}$, then $\alpha w \in \text{Span}(S)$

$w \in \text{Span}(S)$, so

$$w = \sum_{i=1}^{n} \alpha_i v_i \text{ for } \alpha_i \in \mathbb{F},$$

$v_i \in S$ for $1 \leq i \leq n$.

$$\text{Then } \alpha w = \sum_{i=1}^{n} (\alpha \alpha_i) v_i$$

$\in \text{Span}(S)$ by definition.

Then span(S) is a subspace by the subspace test. $\square$